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Relationships between Convolution and Correlation for Fourier Transform and Quaternion Fourier Transform

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Abstract

In this paper we introduce convolution theorem for the Fourier transform (FT) of two complex functions. We show that the correlation theorem for the FT can be derived using properties of convolution. We develop this idea to derive the correlation theorem for the quaternion Fourier transform (QFT) of the two quaternion functions.

Keywords: Fourier transform, quaternion Fourier transform, quaternion convolution, quaternion correlation

I. Introduction

The Fourier transform (FT) plays an important part in the theory of many branches of science and engineering. In the field of applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. In computer vision, images in the spatial domain can be transformed into the frequency domain by the Fourier transform. It is a very useful technique in image processing, because some operations and measurements can be done better in the frequency domain than in the spatial domain [4]. The most fundamental and important properties of the FT are convolution and correlation. They are mathematical operations which have been widely used in the theory of linear time-invariant (LTI) systems.

As a generalization of the FT, the quaternion Fourier transform (QFT), is first proposed by Ell [2]. Later, some constructive works related to QFT and its application in color image processing are presented in [1,3,9]. Recently, some author (see, for example, [5, 6, 7, 12]) have extensively studied the QFT and its properties from a mathematical point of view. They found that most of the properties of this extended transform are generalizations of the corresponding properties of the FT with some modifications.

The purpose of this paper is to show the relationships between the convolution and correlation for the FT and the QFT. We first derive correlation theorem for the FT by applying the properties of the convolution theorem of two complex functions. Similar to the FT case, this approach can be developed to obtain the correlation theorem for QFT of the two quaternion functions.

2. Preliminaries

In this section we briefly review some basic ideas on quaternions. For a more complete discussion we refer the readers to [1].

2.1 Quaternion algebra

The first concept of quaternions, which is a type of hypercomplex number, was formally introduced by Hamilton in 1843 and is denoted by $\mathbb{H}$. It is an associative non-commutative four dimensional algebra

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3\}, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R}. \tag{1}$$

The orthogonal imaginary units $\boldsymbol{i}, \boldsymbol{j},$ and $\boldsymbol{k}$ should follow the multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1,\ \boldsymbol{ij} = -\boldsymbol{ji},\ \boldsymbol{ik} = -\boldsymbol{ki},\ \boldsymbol{jk} = -\boldsymbol{kj}, \text{and } \boldsymbol{ijk} = -1.$$

We may express a quaternion q as a scalar part denoted by $Sc(q) = q_0$ and a pure quaternion $\boldsymbol{q}$ denoted by $\mathrm{Vec}(q) = \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 = \boldsymbol{q}$. The conjugate of a quaternion q is obtained by changing the signs of the pure quaternion, that is, $\bar{q} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3.$

It is a linear anti-involution, that is, for every $p, q \in \mathbb{H}$ we have

$$\bar{\bar{p}} = p, \qquad \overline{p+q} = \bar{p} + \bar{q},\ \overline{pq} = \bar{q}\bar{p}. \tag{2}$$

It is not difficult to see that from equation (1) and the third term of equation (2) we obtain the norm of a quaternion q as

$$|q| = \sqrt{q\bar{q}} = \sqrt{{q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2}.$$

We further get the inverse

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

This fact shows that $\mathbb{H}$ is a normed division algebra. Any quaternion q can be written as

$$q = |q|e^{\mu\theta},$$

where $\theta = \arctan\frac{|\mathrm{Sc}(q)|}{\mathrm{Vec}(q)}$, $0 \leq \theta \leq \pi$ is the eigen angle (phase). When $|q| = 1$, q is a unit quaternion. Euler's and De Moivre's formulas still hold in quaternion space, i.e., for a pure unit quaternion μ the following holds:

$$e^{\mu\theta} = \cos(\theta) + \mu\sin(\theta)$$

$$e^{\mu n\theta} = (\cos(\theta) + \mu\sin(\theta))^n = \cos(n\theta) + \mu\sin(n\theta).$$

2.2 Basic Properties of Fourier Transform

In this section we briefly review the definition of the Fourier transform (FT) and its basic properties.

Definition 2.1 (Fourier Transform) *Let f be in* $L^2(\mathbb{R})$. *Then Fourier transform of complex function f is defined by*

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx. \tag{3}$$

Since $e^{i\omega x} = \cos\omega x + i\sin\omega x$, the above equation can be written in the following form

$$\mathcal{F}\{f\}(\omega) = \int_{-\infty}^{\infty} f(x)\cos\omega x\,dx + i\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx.$$

In the following we collect some basic properties of the FT, which will be used in the next section.

a. Linearity

   If two complex functions $f, g \in L^2(\mathbb{R})$ *and* α, β *are any two complex constants, then*

$$\mathcal{F}\{\alpha f + \beta g\}(\omega) = \alpha\mathcal{F}\{f\}(\omega) + \beta\mathcal{F}\{g\}(\omega). \tag{4}$$

b. Translation

   If the function $g \in L^2(\mathbb{R})$ *and translation of* g *is defined by* $\tau_a g(x) = g(x-a)$, *then*

$$\mathcal{F}\{\tau_a g\}(\omega) = e^{-i\omega\alpha}\mathcal{F}\{f\}(\omega). \tag{5}$$

   Suppose that $f \in L^2(\mathbb{R})$ *and* $\omega_0 \in \mathbb{R}$.. *If the modulation of the function f is defined by* $\mathbb{M}_{\omega_0}f(x) = e^{i\,\omega_0\,x}f(x)$, *then*

$$\mathcal{F}\left\{\mathbb{M}_{\omega_0}f\right\}(\omega) = \mathcal{F}\{f\}(\omega-\omega_0). \tag{6}$$

c. Time-frequency Shift

   The composition of the translation and modulation is called time-frequency shift, i.e.,

$$\mathcal{F}\left\{\mathbb{M}_{\omega_0}f\right\}(\omega) = \mathcal{F}\{f\}(\omega-\omega_0).$$

d. Scaling
Let $f \in L^2(\mathbb{R})$ *and* $a \in \mathbb{R}, a \neq 0$. *If the dilation operator is denoted by* $D_a f(x) = f(ax)$, *then*

$$\mathcal{F}\{D_a\}(\omega) = \frac{1}{|a|}\mathcal{F}\{f\}\left(\frac{\omega}{a}\right) \tag{7}$$

Theorem 2.2 (Inversion Formula) Suppose that $f \in L^2(\mathbb{R})$ *and* $\mathcal{F}\{f\} \in L^1(\mathbb{R})$, *the inverse transform of the function f is given by*

$$\mathcal{F}^{-\iota}[\mathcal{F}\{f\}](x) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{i\omega x}d\omega. \tag{8}$$

3. Convolution and Correlation for Fourier Transform

Two closely-related operations that are very important for signal processing applications are the convolution and correlation theorems. We first define the convolution of two complex functions and its relationship in the FT domain.

Definition 3.1 (Convolution) *Let* $f, g \in L^2(\mathbb{R})$. *The convolution of two complex functions f and* g *is denoted by* $f * g$ *and is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt \tag{9}$$

Theorem 3.2 Suppose that $f, g \in L^2(\mathbb{R})$. *Then the FT of convolution of the functions is given by*

$$\mathcal{F}\{f * g\}(\omega) = \mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega). \tag{10}$$

Theorem 3.3 *Let* $f, g \in L^2(\mathbb{R})$. *Then Fourier transform of* $\tau_a f * g$ *and* $f * \tau_a g$ *is the same and is given by*

$$\mathcal{F}\{\tau_a f * g\}(\omega) = \mathcal{F}\{f * \tau_a g\}(\omega) = e^{-i\omega a}\mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega).$$

Theorem 3.4 *For every* $f, g \in L^2(\mathbb{R})$, *we have*

$$\mathcal{F}\{f * \mathbb{M}_{\omega_0} g\}(\omega) = \mathcal{F}\{f\}(\omega)\mathcal{F}\{g\}(\omega - \omega_0),$$

and

$$\mathcal{F}\{\mathbb{M}_{\omega_0} f * g\}(\omega) = \mathcal{F}\{f\}(\omega - \omega_0)\mathcal{F}\{g\}(\omega). \tag{11}$$

Definition 3.5 (Correlation) *Let* $f, g \in L^2(\mathbb{R})$, *the correlation of two complex functions f and g is defined by the integral*

$$(f \circ g)(x) = \int_{-\infty}^{\infty} f(x+y)\overline{g(y)}dy. \tag{12}$$

Theorem 3.6 *If* $f, g \in L^2(\mathbb{R})$, *the FT of correlation of the two functions defined in (12) is given by*

$$\mathcal{F}\{f \circ g\}(\omega) = \mathcal{F}\{f\}(\omega)\,\overline{\mathcal{F}\{g\}(\omega)} \tag{13}$$

Proof. From the definition of the FT (1) we easily obtain

$$\mathcal{F}\{f \circ g\}(\omega) = \int_{-\infty}^{\infty}(f \circ g)e^{-i\omega x}\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+y)\overline{g(y)}dy\; e^{-\omega x}dx.$$

Changing variable $x+y=v$ in the above expression, we immediately get

$$\begin{aligned}\mathcal{F}\{f\circ g\}(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\overline{g(y)}e^{-i\omega(v-y)}dydv\\ &= \int_{-\infty}^{\infty} f(v)e^{-i\omega v}\int_{-\infty}^{\infty}\overline{g(y)}e^{i\omega y}dydv\\ &= \int_{-\infty}^{\infty} f(v)e^{-i\omega y}dv\int_{-\infty}^{\infty}\overline{g(y)}e^{i\omega v}dy.\end{aligned}$$

Observe that

$$\int_{-\infty}^{\infty}\overline{g(y)}\,e^{i\omega y}dy = \overline{\int_{-\infty}^{\infty} g(y)e^{-i\omega y}dy} = \overline{\mathcal{F}\{g\}(\omega)}.$$

It means that the above expression reduces to

$$\mathcal{F}\{f\circ g\}(\omega) = \mathcal{F}\{f\}(\omega)\,\overline{\mathcal{F}\{g\}(\omega)}.$$

This proves the proof of theorem.

The alternative proof of Theorem 3. 6 is given as follows. Applying the definition of the FT correlation and then taking $y=-u$ we immediately obtain

$$\begin{aligned}(f\circ g)(x) &= \int_{-\infty}^{\infty} f(x+y)\overline{g(y)}dy\\ &= \int_{-\infty}^{\infty} f(x-u)\overline{g(-u)}\,du.\end{aligned}$$

Setting $\overline{g(-u)}=h(u)$ and applying the FT convolution theorem, above yields

$$\begin{aligned}(f\circ g)(x) &= \int_{-\infty}^{\infty} f(x-u)h(u)\,du\\ &= (f*h)(x)\\ &= \mathcal{F}^{-\iota}\,[\mathcal{F}\{f\}(\omega)\mathcal{F}\{h\}(\omega)](x)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}\{f\}(\omega)\mathcal{F}\{h\}(\omega)e^{i\omega x}d\omega.\end{aligned}$$

We remember that

$$\mathcal{F}\{h\}(\omega) = \int_{-\infty}^{\infty}\overline{g(-u)}\,e^{-i\omega u}du = \overline{\int_{-\infty}^{\infty} g(-u)e^{\iota\omega u}} = \overline{\mathcal{F}\{g\}(\omega)}\,.$$

It means that we have

$$(f\circ g)(x) = \mathcal{F}^{-\iota}\left[\mathcal{F}\{f\}(\omega)\overline{\mathcal{F}\{g\}(\omega)}\right](x).$$

As desired.

We next investigate some properties of the relationship between the correlation and the FT. We first establish the relationship between the conjugation of the correlation and its FT.

Theorem 3.7 (**Conjugation Correlation**) *Let* $f,g\in L^2(\mathbb{R})$. *The FT of correlation of the functions is given by*

$$\mathcal{F}\{\overline{f\circ g}\}(\omega) = \overline{\mathcal{F}\{f\}(-\omega)}\,\mathcal{F}\{g\}(-\omega). \qquad (14)$$

Proof. Applying the definition of the FT gives

$$\begin{aligned}\mathcal{F}\{\overline{f\circ g}\}(\omega) &= \int_{-\infty}^{\infty}(\overline{f\circ g})e^{-i\omega x}\,dx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(x+y)}g(y)\,dy\,e^{-i\omega x}dx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(x+y)}g(y)\,e^{-i\omega x}dydx.\end{aligned}$$

Performing the change of variable $x+y=v$ yields

$$\begin{aligned}\mathcal{F}\{\overline{f\circ g}\}(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(v)}g(y)e^{-i\omega(v-y)}dydv\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{f(v)}g(y)e^{-i\omega v}e^{i\omega y}dydv\end{aligned}$$

$$= \int_{-\infty}^{\infty} \overline{f(v)} e^{-i\omega v} dv \int_{-\infty}^{\infty} g(y) e^{i\omega y} dy$$
$$= \int_{-\infty}^{\infty} \overline{f(v) e^{i\omega v}} dv \int_{-\infty}^{\infty} g(y) e^{i\omega y} dy$$
$$= \overline{\mathcal{F}\{f\}(-\omega)}\, \mathcal{F}\{g\}\,(-\omega).$$

Theorem 3.8 *Let* $f, g \in L^2(\mathbb{R})$. *Then we get*

$$\mathcal{F}\{\tau_a f \circ g\}(\omega) = e^{i\omega a} \mathcal{F}\{f\}(\omega) \overline{\mathcal{F}\{g\}(\omega)},$$

and

$$\mathcal{F}\{f \circ \tau_a g\}(\omega) = e^{-i\omega a} \mathcal{F}\{f\} \overline{\mathcal{F}\{g\}(\omega)}. \tag{15}$$

Proof. For the first term of (15) we apply equation (5) and Theorem 3.6 to get

$$\begin{aligned} \mathcal{F}\{\tau_a f \circ g\}(\omega) &= \mathcal{F}\{\tau_a f\}(\omega) \overline{\mathcal{F}\{g\}(\omega)} \\ &= e^{i\omega a} \mathcal{F}\{f\}(\omega) \overline{\mathcal{F}\{g\}(\omega)}. \end{aligned}$$

Theorem 3.9 *For every* $f, g \in L^2(\mathbb{R})$, *we have*

$$\mathcal{F}\{\mathbb{M}_{\omega_0} f \circ g\}(\omega) = \mathcal{F}\{f\}(\omega - \omega_0) \overline{\mathcal{F}\{g\}(\omega)},$$

and

$$\mathcal{F}\{f \circ \mathbb{M}_{\omega_0} g\}(\omega) = \mathcal{F}\{f\}(\omega)\, \overline{\mathcal{F}\,\{g\}(\omega - \omega_0)}. \tag{16}$$

Proof. For the first term of (16), an application of Theorem 3.6 and equation (6) we easily obtain

$$\begin{aligned} \mathcal{F}\{\mathbb{M}_{\omega_0} f \circ g\}(\omega) &= \mathcal{F}\{\mathbb{M}_{\omega_0} f\}(\omega)\, \overline{\mathcal{F}\{g\}(\omega)} \\ &= \mathcal{F}\{f\}(\omega - \omega_0) \overline{\mathcal{F}\{g\}(\omega)}, \end{aligned}$$

which was to be proved.

4. Convolution and Correlation for Quaternion Fourier Transform

In this section we give a definition of the quaternion Fourier transform (QFT) and we then establish the correlation theorem fort the QFT via the properties of the convolution theorem of two quaternion functions.

Definition 4.1 (**QFT**) *Let f be in* $L^2(\mathbb{R}^2; \mathbb{H})$. *Then Quaternion Fourier transform of the function f is given by*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\boldsymbol{x}) e^{-\mu \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d\boldsymbol{x},$$

where μ *is any pure unit quaternion such that* $\mu^2 = 1$, $\boldsymbol{\omega}, \boldsymbol{x} \in \mathbb{R}^2$ *and* $\boldsymbol{\omega} \cdot \boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2$.

Theorem 4.2 (Inverse QFT) *Suppose that* f *be in* $L^2(\mathbb{R}^2;\mathbb{H})$ *and* $\mathcal{F}_q\{f\}\in L^1(\mathbb{R}^2;\mathbb{H})$. *Then inverse transform of the QFT is given by*

$$f(\boldsymbol{x}) \;=\; \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}_q\{f\}(\boldsymbol{\omega})e^{\mu\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d\boldsymbol{\omega}.$$

Definition 4.3 (Quaternion Convolution) *Let* $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$. *The convolution of two quaternion functions f and* g *is denoted by* $f *_q g$ *and isgiven by*

$$\left(f *_q g\right)(\boldsymbol{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{t})g(\boldsymbol{x}-\boldsymbol{t})d\boldsymbol{t}. \tag{17}$$

Based on the definition of the quaternion convolution we obtain the definition of quaternion correlation as follows.

Definition 4.4 (Quaternion Correlation) *Let* $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$ *be two quaternion functions. The correlation of f and* g *is defined by*

$$\left(f \circ_q g\right)(\boldsymbol{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x}+\boldsymbol{y})\overline{g(\boldsymbol{y})}d\boldsymbol{y}. \tag{18}$$

We have the following result (compare to [9]).

Theorem 4.4 Suppose that $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$. *Then the QFT of convolution of the functions is given by*

$$\begin{aligned}\mathcal{F}_q\{f *_q g\}(\omega) &= \mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \boldsymbol{i}\,\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\\ &\quad+\boldsymbol{j}\,\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega}) + \boldsymbol{k}\,\mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega}).\end{aligned}$$

In order to study the relationship between the quaternion convolution and quaternion correlation we derive the following result by using Theorem 4.4.

Theorem 4.5 Suppose that $f,g\in L^2(\mathbb{R}^2;\mathbb{H})$. *Then the QFT of correlation of the functions is given by*

$$\begin{aligned}\mathcal{F}_q\{f \circ_q g\}(\omega) &= \mathcal{F}_q\{g_0\}(-\boldsymbol{\omega}) - \mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega})\\ &\quad+\boldsymbol{i}\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})\,\mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\,\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\\ &\quad+\boldsymbol{j}\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega}) - \mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega})\\ &\quad+\boldsymbol{k}\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})\mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\,\mathcal{F}_q\{f_1\}(\boldsymbol{\omega}).\end{aligned}$$

Proof. Proceeding as in the alternative proof of Theorem 3. 2 we immediately get

$$\begin{aligned}\left(f \circ_q g\right)(\boldsymbol{x}) &= \int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x}+\boldsymbol{y})\overline{g(\boldsymbol{y})}d\boldsymbol{y}\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x}-\boldsymbol{u})\overline{g(-u)}\,d\boldsymbol{u}.\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\boldsymbol{x}-\boldsymbol{u})h(\boldsymbol{u})\,d\boldsymbol{u}\\ &= \left(f *_q h\right)(\boldsymbol{x})\\ &= \mathcal{F}_q^{-1}[\mathcal{F}_q\{h\}(\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \boldsymbol{i}\,\mathcal{F}_q\{h\}(\boldsymbol{\omega})\mathcal{F}_q\{f_1\}(\boldsymbol{\omega})\\ &\quad+\boldsymbol{j}\,\mathcal{F}_q\{h\}(\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega}) + \boldsymbol{k}\,\mathcal{F}_q\{h\}(\boldsymbol{\omega})\mathcal{F}_q\{f_3\}(\boldsymbol{\omega})]\,(\boldsymbol{x}).\end{aligned}$$

On the other hand, by simple computation we get

$$\begin{aligned}\mathcal{F}_q\{h\}(\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{g(-u)}\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{u}} d\boldsymbol{u} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g_0(-\boldsymbol{u}) - \boldsymbol{g}(-\boldsymbol{u}))\, e^{-\mu\boldsymbol{\omega}\cdot\boldsymbol{u}} d\boldsymbol{u} \\ &= \mathcal{F}_q\{g_0\}(-\boldsymbol{\omega}) - \mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega}),\end{aligned} \tag{19}$$

where $\boldsymbol{g}(-\boldsymbol{u}) = \boldsymbol{i}g_1(-\boldsymbol{u}) - \boldsymbol{j}g_2(-\boldsymbol{u}) - \boldsymbol{k}g_3(-\boldsymbol{u})$. Substituting this fact into the right-hand side of (19) we finally obtain

$$\begin{aligned}(f \circ_q g)(\boldsymbol{x}) = \mathcal{F}_q^{-1}[&\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega}) - \mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) \\ &+\boldsymbol{i}\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})\, \mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\, \mathcal{F}_q\{f_1\}(\boldsymbol{\omega}) \\ &+\boldsymbol{j}\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega}) - \mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\mathcal{F}_q\{f_2\}(\boldsymbol{\omega}) \\ &+\boldsymbol{k}\mathcal{F}_q\{g_0\}(-\boldsymbol{\omega})\mathcal{F}_q\{\boldsymbol{g}\}(-\boldsymbol{\omega})\, \mathcal{F}_q\{f_1\}(\boldsymbol{\omega})](\boldsymbol{x}).\end{aligned}$$

This is the desired result.

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